Gauss-Bonnet lagrangian $G \ln G$ and cosmological exact solutions

Hans-Jürgen Schmidt

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Institut für Mathematik, Universität Potsdam, Germany
Am Neuen Palais 10, D-14469 Potsdam, hjschmi@rz.uni-potsdam.de

Abstract

For the lagrangian $L = G \ln G$ where G is the Gauss-Bonnet curvature scalar we deduce the field equation and solve it in closed form for 3-flat Friedman models using a statefinder parametrization.

Further we show, that among all lagrangians $F(G)$ this L is the only one not having the form G^r with a real constant r but possessing a scale-invariant field equation. This turns out to be one of its analogies to $f(R)$-theories in 2-dimensional space-time.

In the appendix, we systematically list several formulas for the decomposition of the Riemann tensor in arbitrary dimensions n, which are applied in the main deduction for $n = 4$.

1 Introduction

Fourth-order gravity has been a serious alternative to General Relativity since 1918 already: H. Weyl, see [1], was guided by the idea of the scale invariance of the action which requested for an R^2-term in its integrand instead of the Einstein-Hilbert action integrand R. In fact, the integral $\int R^n \sqrt{-g} d^k x$ in k-dimensional space-time is scale-invariant just in the case $k = 2n$, leading to $n = 2$ for the usual space-time dimension $k = 4$. For details see e.g. the reviews [2], [3], [4], and [5], and the books [6] and [7]. For a broader view to this topic, and also on the growth of (quantum) perturbations to the today's observed large-scale structures by inflation, see the references cited there.

Since 1947 it became clearer, that the cosmological evolution can be better modeled if both R and R^2-terms belong to the action, see C. Gregory [8]. In the eighties, the inflationary cosmology has been related to fourth-order gravity by Starobinsky [9], and this paper initiated several follow-up papers, e.g. [10] and [11]; generalizations by inclusion of R^3-terms and later by a general $f(R)$ have been worked out e.g. in [12] and [13].

In 1921, R. Bach [14], see [15] for details, initiated a detailed investigation of the conformally invariant field equations following from the lagrangian $C_{ijkl}C^{ijkl}$. In 1977 it was shown, that a theory with lagrangian of the form

$$\Lambda + R + \alpha R^2 + \beta C_{ijkl}C^{ijkl} \tag{1.1}$$

where units are chosen that light velocity equals 1 and Newton's constant equals $(16\pi)^{-1}$, can be renormalized, see K. Stelle [16].

In this context it is often mentioned, that the addition of a multiple of the Gauss-Bonnet term, see also [17] and [18],

$$G = R_{ijkl}R^{ijkl} - 4R_{ij}R^{ij} + R^2 \tag{1.2}$$

to a lagrangian like (1.1) does not alter the field equations, but it leads to a surface term which may become essential in the quantization. The relation to topology is as follows: the field equations come out by applying continuous deformations of the metric, but $\int G\sqrt{-g}d^4x$ is a topological invariant, see [19], [20], [21], and [22]. Lanczos deduced only the 4-dimensional case, whereas Lovelock generalized to arbitrary dimensions. His sequence L_n starts with $L_1 = R$, $L_2 = G$, and each L_n leads to a topological invariant in the $2n$-dimensional space or space-time.

Recently, a lot of papers appeared which contain the Gauss-Bonnet term in the action. To circumvent the vanishing of its variational derivative, essentially three ways have been gone: Models in dimension larger than 4, see e.g. [23], [24], [25], [26], [27], and [28], models where G is multiplied by a scalar ϕ, see [29], [30], and models where $F(G)$ instead of G is used in the lagrangian with a suitably chosen non-linear function F, see e.g. [31]. Applications of theories with Gauss-Bonnet term to cosmology can also be found in [32], [33], [34], [35], [36], [37], [38], [39], [40], [41], [42], [43], and [44].

2 Statefinder parametrization

The metric of a 3-flat Friedman model with synchronized time coordinate t reads

$$ds^2 = dt^2 - a^2(t)\left(dx^2 + dy^2 + dz^2\right), \qquad a(t) > 0\,. \tag{2.1}$$

We assume that the Taylor development of $a(t)$ exists, the dot denotes d/dt, and the Hubble parameter h is defined as usual via

$$h(t) = \frac{\dot{a}}{a} = \dot{\alpha}\,, \qquad \alpha = \ln a\,. \tag{2.2}$$

In what follows, we always exclude a constant function $a(t)$ as it represents the trivial Minkowski space-time solution. So, we restrict to functions $a(t)$ which have $h(t) = 0$ at isolated moments of time only; at those moments, our exact solutions to be deduced below, have to be matched together.

A time-inversion leads to a change of the sign of h, so we may assume in the following, that always $h(t) > 0$. Under these circumstances we define for any natural number $n \geq 2$

$$z_n = \frac{a^{(n)}\, a^{n-1}}{\dot{a}^n}\,, \qquad a^{(n)} = \frac{d^n a}{dt^n}\,. \tag{2.3}$$

This expression is, up to a constant factor, uniquely determined by the conditions that it is proportional to the n-th time derivative of a with proportionality factor containing a and $\dot{a}$ only, is time-reparametrization invariant, and is scale-invariant. By obvious reasons it proves useful to define also $z_0 = z_1 = 1$.

Another, but equivalent method to define the parameters z_n goes as follows: it is the only product of $a^{(n)}$ with powers of a and $\dot{a}$ which is dimensionless in both interpretations of metric (2.1). In the first interpretation of (2.1), a is a dimensionless quantity, dimensions are encoded in t, x, y and z. In the second interpretation of (2.1), x, y and z are dimensionless quantities, dimensions are encoded in t and a; t is being measured in seconds.

These parameters z_n are especially useful if one wants to solve a scale-invariant field equation as we are going to do below. As usual, we define a field equation to be scale-invariant if for any of its solutions g_{ij} and any real constant c, also the homothetically related metric $e^{2c}\, g_{ij}$ represents a solution.

Our parameters z_n are related to the more usual notation, see [45], [46], [47], [48], and [49], as follows:

$$z_2 = \frac{\ddot{a}\, a}{\dot{a}^2} = -q\,, \quad z_3 = j\,, \quad z_4 = -k\,. \tag{2.4}$$

Here, q is the deceleration parameter, j the jerk parameter, and k the kerk parameter. The notion statefinder parameter refers to the pair $(j,\, s)$, where s is defined for $q \neq 1/2$ via

$$s = \frac{j-1}{3(q-1/2)}\,. \tag{2.5}$$

Next, we give some relations between the parameters z_n and the Hubble parameter h: Solving eq. (2.3) for $a^{(n)}$ we get

$$a^{(n)} = \frac{z_n\, \dot{a}^n}{a^{n-1}}\,. \tag{2.6}$$

The temporal derivative of eq. (2.6) has the same l.h.s. as eq. (2.6) with n replaced by $n+1$. Equating the related r.h.sides we get

$$z_{n+1} = \dot{z}_n/h + z_n\,(n\, z_2 + 1 - n)\,. \tag{2.7}$$

From eqs. (2.2) and (2.4) we easily deduce z_2 in dependence of h:

$$z_2 = -q = 1 - \frac{d}{dt}\left(\frac{1}{h}\right) = 1 + \dot{h}/h^2\,, \tag{2.8}$$

and for $n \geq 3$ we can iteratively deduce z_n in dependence of h with eq. (2.7), the next two terms being

$$z_3 = j = \frac{\dot{z}_2}{h} + z_2(2z_2 - 1) = 1 + 3\dot{h}/h^2 + \ddot{h}/h^3 \tag{2.9}$$

and

$$z_4 = -k = 1 + \frac{6\dot{h}}{h^2} + \frac{4\ddot{h}}{h^3} + \frac{3\dot{h}^2}{h^4} + \frac{1}{h^4}\frac{d^3h}{dt^3}\,.$$

So we get the relation to the other set of dimensionless constants, see eq. (3.12) of [2]

$$\varepsilon_p = \frac{d^p h}{dt^p} \cdot h^{-p-1}\,. \tag{2.10}$$

This leads to $z_1 = \varepsilon_0 = 1$, $z_2 = 1 + \varepsilon_1$, and $z_3 = 1 + 3\varepsilon_1 + \varepsilon_2$.

If we take the logarithmic cosmic scale factor α (see eq. (2.2)) as new time coordinate we can rewrite eq. (2.7) as

$$z_{n+1} = \frac{dz_n}{d\alpha} + z_n \left(n\, z_2 + 1 - n\right). \tag{2.11}$$

After some reformulation we also get as metric

$$ds^2 = \frac{d\alpha^2}{h(\alpha)^2} - e^{2\alpha}\left(dx^2 + dy^2 + dz^2\right) \tag{2.12}$$

with the scale invariant parameters being

$$q = -1 - \frac{1}{h} \cdot \frac{dh}{d\alpha}\,, \tag{2.13}$$

from eq. (2.8), and from eqs. (2.9.) and (2.10)

$$j = 2q^2 + q - \frac{dq}{d\alpha} \tag{2.14}$$

and

$$k = 3jq + 2j - \frac{dj}{d\alpha}\,. \tag{2.15}$$

We will apply these formulas in section 4.

3 Gauss-Bonnet lagrangian

For 2-dimensional space-times, lagrangians of the type $f(R)$ have been discussed e.g. in [51], [52], [53], and [54]. In [51], the lagrangian

$$f(R) = R^{k+1} \tag{3.1}$$

was shown to lead to non-trivial classical results even in the limit $k \to 0$. In [53], this limit was shown to produce the same field equation as the lagrangian

$$f(R) = R \cdot \ln R\,. \tag{3.2}$$

This property is related to the fact that $\int R\sqrt{g}d^2x$ is a topological invariant related to the genus of the space.

Similarly, in [54], the integrand R was kept constant, but instead, the dimension of space-time was formally defined as $2 + \epsilon$, and the limit $\epsilon \to 0$ was discussed.

We now want to transfer this idea to the set of 4-dimensional space-times. To this end we consider a general function $F(G)$ with G from eq. (1.2) as integrand of the action. The full field equations are given e.g. in eq. (5) of [35] using the notation $F_G = dF(G)/dG$,

$$\begin{aligned}
0 = \frac{1}{2} g^{ij} F(G) - 2F_G R R^{ij} + 4F_G R^i_k R^{kj} - 2F_G R^{iklm} R^j{}_{klm} \\
-4F_G R^{iklj} R_{kl} + 2RF_G^{;ij} - 2g^{ij} R \Box F_G - 4R^{ik} F_{G;k}^{;j} \\
-4R^{jk} F_{G;k}^{;i} + 4R^{ij} \Box F_G + 4g^{ij} R^{kl} F_{G;kl} - 4R^{ikjl} F_{G;kl} \,.
\end{aligned} \tag{3.3}$$

The integral $I_G = \int G\sqrt{-g} d^4x$ is a topological invariant related to the Euler characteristic. Therefore, the function $F(G) = G$ leads to the field equation reading $0 = 0$ trivially fulfilled by all metrics g_{ij}, i.e., every space-time represents a stationary point of the action I_G.

Next, we consider the action

$$I = \int F(G)\sqrt{-g} d^4x \tag{3.4}$$

and ask for its properties if a scale transformation is applied to the metric. More exactly: What happens with I eq. (3.4) if we replace g_{ij} by its homothetically equivalent metric $e^{2\gamma} g_{ij}$ where γ is an arbitrary constant? If I does not change at all by this transformation, then we call I scale-invariant. G goes over to $e^{-4\gamma} G$ by this transformation, and so, obviously, only $F(G) = c_2 \cdot G$ with a constant c_2 leads to a scale-invariant action I.

A less trivial question is the following one: Under which conditions, the action I eq. (3.4) is almost scale-invariant, i.e., scale-invariant up to adding a multiple of I_G? In other words: Which functions $F(G)$ have the property that replacing g_{ij} by $e^{2\gamma} g_{ij}$ in eq. (3.4) leads to the action $I + k_\gamma \cdot I_G$ with constant k_γ? The answer is: Besides the case already discussed above,

$$F(G) = c_1 \cdot G \cdot \ln G + c_2 \cdot G \tag{3.5}$$

with constants $c_1 \neq 0$ and c_2 is the complete set of solutions.[1] A word to dimensions: the argument of the logarithm should be dimensionless, so, instead of $\ln G$ we should have written $\ln(G/G_0)$. However, a change of the value G_0 can be compensated by a redefinition of the constant c_2. As the term with c_2 does not contribute to the field equation, we may put it to

[1]For negative values of G, the term $\ln G$ should be replaced by $\ln |G|$. The singularity at $G \to 0$ is a mild one and in the models of our interest, $|G|$ is positive anyhow.

zero classically. Dividing everything by c_1 we finally get the only interesting remaining almost scale-invariant case to be

$$F(G) = G \cdot \ln G \,. \tag{3.6}$$

If we insert eq. (3.6) into eq. (3.3), the following simpler field equation appears:

$$\begin{aligned} 0 = \frac{1}{2} g^{ij} G \cdot \ln G - 2(RR^{ij} - 2R^i_k R^{kj} + R^{iklm} R^j{}_{klm} \\ +2R^{iklj} R_{kl}) \cdot (1 + \ln G) + 2R(\ln G)^{;ij} - 2g^{ij} R \Box (\ln G) - 4R^{ik} (\ln G)^{;j}_{;k} \\ -4R^{jk}(\ln G)^{;i}_{;k} + 4R^{ij} \Box (\ln G) + 4g^{ij} R^{kl} (\ln G)_{;kl} - 4R^{ikjl} (\ln G)_{;kl} \,. \end{aligned} \tag{3.7}$$

Due to its importance it seems justified to deduce this case by another way: Take a small positive parameter ϵ and define

$$F_\epsilon(G) = \frac{1}{\epsilon} \cdot \left(G^{1+\epsilon} - G \right) \tag{3.8}$$

which leads to the same vacuum equation as the lagrangian $G^{1+\epsilon}$. Then the limit $\epsilon \to 0$ in eq. (3.8) exactly leads to (3.6). Sketch of the proof: Put $G = e^x$, then

$$G^\epsilon = e^{\epsilon x} \approx 1 + \epsilon x = 1 + \epsilon \ln G \,.$$

4 Exact Friedman models

We apply the notation of section 2, especially metric (2.1) with Hubble parameter (2.2) etc. If we start with $a(t) = t^n$ with positive values n and t, we get $h = n/t$, $\alpha = n \ln t$, $q = (1-n)/n$, $j = (n-1)(n-2)/n^2$, $k = -(n-1)(n-2)(n-3)/n^3$, $t = e^{\alpha/n}$, and $h(\alpha) = n \cdot e^{-\alpha/n}$. The metric can then also be written as

$$ds^2 = \frac{d\alpha^2}{n^2} e^{2\alpha/n} - e^{2\alpha} \left(dx^2 + dy^2 + dz^2 \right) \,. \tag{4.1}$$

This leads to the de Sitter space-time as $n \to \infty$ where $q = -1$, $j = 1$, $k = -1$, and $s = 0$. Within Einstein's theory and with pressureless matter of density ρ, the deceleration q is related to the critical density ρ_c necessary to close the universe via $2q = \rho/\rho_c$.

Using eqs. (6) and (7) of [35], or using eq. (3) of [38] we get

$$R = 6 \left(\dot h + 2h^2 \right) \,, \qquad G = 24 \left(\dot h h^2 + h^4 \right) \tag{4.2}$$

and the vacuum equation following from the action (3.4) as

$$0 = G \cdot F_G - F(G) - 24\dot{G} \cdot h^3 \cdot F_{GG}\,, \tag{4.3}$$

where $F_G = dF/dG$ and $F_{GG} = dF_G/dG$. In comparison with the full field equation (3.3), this is a surprisingly simple equation. We test the previously discussed property as follows: adding $c_2 \cdot G$ to this F, the set of solutions to eq. (4.3) will not change. For non-vanishing F_{GG}, i.e., a non-linear function $F(G)$, eq. (4.3) is of third order in the metric, as it represents the constraint equation to the full fourth-order field equation.

Now we insert the example $F(G) = G \ln G$ of eq. (3.6) into eq. (4.3) and get via $F_G = 1 + \ln G$ and $F_{GG} = 1/G$ and after multiplication with $G = -24h^4 \cdot q$

$$0 = G^2 - 24\dot{G} \cdot h^3\,. \tag{4.4}$$

The singular case $G = 0$ needs an extra consideration: Looking at eq. (4.2) this leads to $\dot{h} = -h^2$, as the case $h = 0$ was already excluded earlier. This behaviour can be written in the original form (2.1) by $a(t) = t$, i.e. the deceleration vanishes identically: $q = 0$.

Now we look for the remaining solutions of eq. (4.4), i.e., those with $G \neq 0$. To this end we insert eqs. (2.8) and (4.2) into eq. (4.4). The result is the second order equation for h

$$\left(\dot{h}h^2 + h^4\right)^2 = h^3 \cdot \frac{d}{dt}\left(\dot{h}h^2 + h^4\right) \tag{4.5}$$

reducing via $\frac{\dot{q}}{h} = \frac{dq}{d\alpha}$, see eq. (2.2), to the following first-order equation for the deceleration parameter q

$$\frac{dq}{d\alpha} = 4q + 3q^2\,. \tag{4.6}$$

The fact, that eq. (4.6) does not contain the Hubble parameter is a consequence of the scale-invariance of the field equation. By the way, eqs. (2.14) and (4.6) can be combined to $q^2 + 3q + j = 0$ characterizing this field equation.

The other solution with constant value q is $q = -4/3$. Using eq. (2.8) we get $3\dot{h} = h^2$, i.e. $h = -3/t$ and finally $a(t) = 1/t^3$. These two solutions with constant q, i.e. $a(t) = t$ and $a(t) = 1/t^3$, represent themselves self-similar space-times: Multiplying the metric of space-time with a constant factor can be compensated by a time-translation.

Let us finally come to the case on non-constant q in eq. (4.6). Considering solutions as same, if they are related by a scale-transformation, exactly three solutions remain, characterized by

$$q(\alpha) = -\frac{4}{3 + 3 \cdot e^{-4\alpha}}, \qquad -\frac{4}{3} < q < 0 \tag{4.7}$$

and

$$q(\alpha) = -\frac{4}{3 - 3 \cdot e^{-4\alpha}}, \tag{4.8}$$

where α may take all real values in eq. (4.7), but eq. (4.8) is not defined for $\alpha = 0$ and represents one solution for $\alpha > 0$, i.e. $q < -4/3$ and another one for $\alpha < 0$, i.e. $q > 0$. Coming back to a relation for the scale factor, we get

$$q(a) = -\frac{\ddot{a}\, a}{\dot{a}^2} = -\frac{4}{3 \pm 3/a^4}. \tag{4.9}$$

One of the two remaining quadratures can still be done in explicit form via eq. (2.13), i.e. $\frac{d(\ln h)}{d\alpha} = -1 - q$, leading to

$$h(a) = \frac{c}{a} \cdot |a^4 \pm 1|^{-1/3} \tag{4.10}$$

with a positive constant c. The final step to get the function $a(t)$ is then via the integral

$$\int_{a(0)}^{a(t)} |a^4 \pm 1|^{-1/3} da = c \cdot t. \tag{4.11}$$

5 Conclusion

For the lagrangian $L = G \ln G$ where G, see eq. (1.2), is the Gauss-Bonnet curvature scalar we deduced the field equation and solved it completely up to one final quadrature eq. (4.11) in closed form for 3-flat Friedman models using a statefinder parametrization. Further we have shown, that among all lagrangians $F(G)$ this L is the only one not having the form G^r with a real constant r but possessing a scale-invariant field equation. This turns out to be one of its analogies to $f(R)$-theories in 2-dimensional space-time.

Recently, several other modifications of Einstein gravity have been discussed, see e.g. [55] for a non-local one, here we propose with the arguments given above, as gravitational lagrangian

$$L_g = \Lambda + R + \alpha R^2 + \beta C_{ijkl} C^{ijkl} + \gamma G \ln G \tag{5.1}$$

being worth considered in more details than done up to now.

6 Appendix

Here we present some decompositions of the Riemann tensor from the geometric point of view which are implicitly used in the text above, and which may have some interest in themselves and have other applications, too.

In four dimensions, the Riemann tensor R_{ijkl} possesses $4^4 = 256$ real components. By use of the known symmetries, this figure reduces to 20, but this twenty-dimensional space is even harder to imagine. Example: To work with the field equation (3.3) it is necessary to know, that in four dimensions,

$$C^{iklm}C_{jklm} = \frac{1}{4}\ \delta^i_j\ C^{gklm}C_{gklm}$$

and how this can be used for evaluating analogous terms with the Riemann tensor.

Below, we will present four different possibilities how to arrange this set of components to get a better understandable system.

The Riemann tensor R_{ijkl} of a space-time of dimension $n \geq 3$ can be decomposed according to several different criteria:
1. The usual one into the Weyl tensor C_{ijkl} plus a term containing the Ricci tensor R_{ij} plus a term containing the Riemann curvature scalar R.
2. Two trace-less parts plus the trace.
3. The Weyl tensor plus only one additional term.
4. Two divergence-free parts plus the trace.

We use the following two properties of the Riemann tensor

$$R_{ijkl} = -R_{ijlk}\,, \qquad R_{ijkl} = R_{klij}\,. \tag{6.1}$$

The Ricci tensor is the trace of the Riemann tensor: $R_{ij} = g^{kl}R_{ikjl}$, where g_{kl} denotes the metric of the space-time, and the Riemann curvature scalar is the trace of the Ricci tensor $R = g^{kl}R_{kl}$. The sign conventions are defined such that in Euclidean signature, the curvature scalar of the standard sphere is positive.

For any symmetric tensor H_{ij} we define another tensor H^*_{ijkl} via

$$H^*_{ijkl} = H_{ik}g_{jl} + H_{jl}g_{ik} - H_{il}g_{jk} - H_{jk}g_{il}\,. \tag{6.2}$$

Then the tensor H^*_{ijkl} automatically fulfils the identities eq. (6.1). For the special case $H_{ij} = g_{ij}$ we get the simplified form

$$g^*_{ijkl} = 2g_{ik}g_{jl} - 2g_{il}g_{jk}\,. \tag{6.3}$$

6.1 The usual decomposition

The Weyl tensor C_{ijkl} is the trace-less part of the Riemann tensor, i.e. $g^{ik} C_{ijkl} = 0$. It vanishes identically for $n = 3$. Using the notation of eqs. (6.2) and (6.3) we make the ansatz

$$R_{ijkl} = C_{ijkl} + \alpha R^*_{ijkl} + \beta R \, g^*_{ijkl} \,. \tag{6.4}$$

Then the coefficients α and β have to be specified such that the trace-lessness condition for the Weyl tensor becomes an identity. This condition determines the coefficients α and β uniquely, and the result is:

$$\alpha = \frac{1}{n-2} \qquad \text{and} \qquad \beta = \frac{-1}{2(n-1)(n-2)} \,. \tag{6.5}$$

Thus, we get the usual formula

$$R_{ijkl} = C_{ijkl} + \frac{1}{n-2} \left(R_{ik} g_{jl} + R_{jl} g_{ik} - R_{il} g_{jk} - R_{jk} g_{il} \right) - \frac{1}{(n-1)(n-2)} R \left(g_{ik} g_{jl} - g_{il} g_{jk} \right) \,.$$

6.2 The decomposition using trace-less parts

In distinction to the previous subsection, we now perform a more consequent decomposition into trace and trace-less parts. To this end we define S_{ij} as the trace-less part of the Ricci tensor, i.e. $g^{ij} S_{ij} = 0$ with $S_{ij} = R_{ij} + \kappa R g_{ij}$ possessing the unique solution $\kappa = -1/n$, i.e., $S_{ij} = R_{ij} - R \, g_{ij}/n$. Then the analogous equation to eq. (6.4) is

$$R_{ijkl} = C_{ijkl} + \gamma S^*_{ijkl} + \delta R \, g^*_{ijkl} \,. \tag{6.6}$$

This becomes a correct identity if and only if

$$\gamma = \frac{1}{n-2} \qquad \text{and} \qquad \delta = \frac{1}{2n(n-1)} \,. \tag{6.7}$$

So we get

$$R_{ijkl} = C_{ijkl} + \frac{1}{n-2} \left(S_{ik} g_{jl} + S_{jl} g_{ik} - S_{il} g_{jk} - S_{jk} g_{il} \right) + \frac{1}{n(n-1)} R \left(g_{ik} g_{jl} - g_{il} g_{jk} \right) \,.$$

6.3 Decomposition into two parts

Let us define a tensor $L_{ij} = R_{ij} + \zeta R\, g_{ij}$ such that a parameter ε exists which makes

$$R_{ijkl} = C_{ijkl} + \varepsilon L^*_{ijkl} \tag{6.8}$$

becoming a true identity. It turns out that this is possible if and only if

$$\zeta = \frac{-1}{2(n-1)} \qquad \text{and} \qquad \varepsilon = \frac{1}{n-2}\,. \tag{6.9}$$

Thus, we can write $L_{ij} = R_{ij} - \frac{1}{2(n-1)} R\, g_{ij}$ and

$$R_{ijkl} = C_{ijkl} + \frac{1}{n-2}\left(L_{ik}g_{jl} + L_{jl}g_{ik} - L_{il}g_{jk} - L_{jk}g_{il}\right)\,. \tag{6.10}$$

6.4 Decomposition into divergence-free parts

Now, besides the identities from eq. (6.1), we also use identities involving the covariant derivatives, denoted by a semicolon, of the Riemann tensor. The Bianchi identity reads

$$R_{ijkl;m} + R_{ijlm;k} + R_{ijmk;l} = 0\,.$$

Its trace can be obtained by transvection with g^{ik} and reads

$$R_{jl;m} + R^i{}_{jlm;i} - R_{jm;l} = 0\,. \tag{6.11}$$

It should be mentioned, that the transvection with respect to other pairs of indices does not lead to further identities. The Einstein E_{ij} tensor is defined as $E_{ij} = R_{ij} + \lambda\, R\, g_{ij}$, where λ has to be chosen such that the Einstein tensor is divergence-free, i.e., $E^i{}_{j;i} = 0$. Using the trace of eq. (6.11) (again, there is essentially only one such trace), namely $2R^i{}_{l;i} - R_{;l} = 0$, we uniquely get $\lambda = -1/2$, i.e. the Einstein tensor is $E_{ij} = R_{ij} - \frac{1}{2} R\, g_{ij}$.

With the ansatz

$$R_{ijkl} = W_{ijkl} + \eta E^*_{ijkl} + \theta R\, g^*_{ijkl} \tag{6.12}$$

it holds: The coefficients η and θ are uniquely determined by the requirements that eq. (6.12) is an identity, and the divergence of the tensor W_{ijkl} vanishes:

${W^i}_{jkl;i} = 0$. We get uniquely the following values of the constants: $\eta = 1$ and $\theta = \frac{1}{4}$. Then

$$
\begin{aligned}
R_{ijkl} = W_{ijkl} + E_{ik} g_{jl} + E_{jl} g_{ik} - E_{il} g_{jk} - E_{jk} g_{il} \\
+ \frac{1}{2} R \left(g_{ik} g_{jl} - g_{il} g_{jk} \right)
\end{aligned}
\tag{6.13}
$$

defines a decomposition of the Riemann curvature tensor into the divergence-free tensors W_{ijkl}, E_{ij}, g_{ij} and the scalar R.

It should be mentioned, that for every $n > 2$, the four tensors R_{ij}, S_{ij}, L_{ij}, and E_{ij} represent four different tensors. And it is a remarkable fact, that the coefficients in E_{ij} and in eq. (6.13) do not depend on the dimension n.

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